Quantum Mechanics inspired decoding algorithm for error-correcting codes

Jun-ichi Inoue
Hokkaido University, Sapporo, JAPAN

In collaboration with
Yohei Saika, Masato Okada
Wakayama NCT, The Univ. of Tokyo

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Aim of the study

For error-correcting codes described by spin glass models (a kind of magnetic alloy), we construct an iterative algorithm to achieve the Bayes-optimal decoding by making use of quantum-mechanical fluctuation.

We evaluate the statistical performance of the algorithm.

Conceptual picture ⇒

We show the usefulness of the quantum-mechanical fluctuation to solve the problem of information.
Outline

• Error-correcting codes (Sourlas codes)
• Bayesian inference and energy function
• Quantum-mechanical fluctuation
• Phase diagram of the performance
• Iterative decoding algorithm
• Summary
Error-correcting codes

Original bit sequence

1010010100101010101

Received bit sequence

1000000100101011101

We send

\[ N \sum_{p} \text{parity bits} \]

The Shannon’s theorem tells us

\[ R < C \] (Channel capacity)

\[ p_e \sim e^{-N(C-R)} \]

Rate

\[ R = \frac{N}{N^C_p} \]

Parity bits

Error-correcting codes

Error

Noisy channel

\[ 1 + 0 + 1 + 0 + 0 = 0 \pmod{2} \]

\[ 0 + 1 + 0 + 1 + 0 = 0 \pmod{2} \]

\[ 0 + 1 + 0 + 0 + 1 = 0 \pmod{2} \]

\[ 1 + 0 + 1 + 0 + 1 = 1 \pmod{2} \]
Error-correcting codes by ‘spins’

Original message (Ising spins)
\[ \{ \xi_1, \xi_2, \ldots, \xi_N \}, \quad \xi \in \{-1, 1\} \]

Parity
\[ \xi_{i1} \xi_{i2} \cdots \xi_{ip} = J_{i1i2 \cdots ip}^0 \]

(# of parity is \( N \binom{p}{C_p} \))

Rate and capacity (\( N, p \to \infty \))
\[ R = \frac{N}{N_B} = \frac{N}{N \binom{p}{C_p}} \approx \frac{p!}{N^{p-1}} \]
\[ C \approx \frac{J_0^2 p!}{J^2 N^{p-1} \log 2} \]

\[ \frac{R}{C} = \left( \frac{J}{J_0} \right)^2 \log 2 \leq 1 \]

S/N ratio
error \sim e^{-N(C-R)}

Noise (AWGN)
\[ J_{i1i2 \cdots ip} = \left( \frac{J_0 p!}{N^{p-1}} \right) J_{i1i2 \cdots ip}^0 + J \sqrt{\frac{p!}{2 N^{p-1}}} \xi_{i1i2 \cdots ip}^0 \]
\[ \xi_{i1i2 \cdots ip} = N(0,1), \quad \left\langle \xi_{i1i2 \cdots ip} \, \xi_{j1j2 \cdots jp} \right\rangle = \delta_{i1i2 \cdots ip, j1j2 \cdots jp} \]
Sourlas codes (1989)

Posterior

\[ P(\{\sigma\} | \{J\}) \propto P(\{J\} | \{\sigma\}) P(\{\sigma\}) \]

Here we assume uniform prior

\[
\frac{\exp\left(-\frac{N^{p-1}}{a^2 p!} \sum_{i1i2..ip} \xi_{i1i2..ip}^2 \right)}{\left( a^2 \pi p! / N^{p-1} \right)^{\frac{1}{2}}} = \frac{\exp\left(-\frac{N^{p-1}}{a^2 p!} \sum_{i1..ip} (J_{i1..ip} - \frac{a_0 p!}{N^{p-1}} \sigma_{i1} \cdots \sigma_{ip})^2 \right)}{\left( a^2 \pi p! / N^{p-1} \right)^{\frac{1}{2}}}
\]

MAP estimation is identical to finding the lowest energy state of

\[ H = \frac{N^{p-1}}{a^2 p!} \sum_{i1..ip} (J_{i1..ip} - \frac{a_0 p!}{N^{p-1}} \sigma_{i1} \cdots \sigma_{ip})^2 \]

\[ J_{i1..ip} \]: Output of AWGN (noisy parity)

Minimization of the energy is achieved via simulated annealing etc
Bayes-optimal decoding

Posterior marginal:

\[
P(\sigma_i \mid \{J\}) = \sum_{\{\sigma\} \neq \sigma_i} P(\{\sigma\} \mid \{J\})
\]

\[
\bar{\xi}_i = \text{sgn} \left[ P(\sigma_i = 1 \mid \{J\}) - P(\sigma_i = -1 \mid \{J\}) \right]
\]

We might rewrite it as

\[
\bar{\xi}_i = \text{sgn} \left[ \sum_{\sigma_i} \sigma_i P(\sigma_i \mid \{J\}) \right] = \text{sgn} \left( \frac{\sum_{\{\sigma\}} \sigma_i e^{-H/T}}{\sum_{\{\sigma\}} e^{-H/T}} \right) = \text{sgn} \left( \langle \sigma_i \rangle_{T=1} \right)
\]

Bit-error rate

\[
p_B = \frac{1}{2} \left( 1 - \frac{1}{N} \sum_i \bar{\xi}_i \xi_i \right)
\]

is minimized on the Nishimori line

\[
\frac{a_0}{a^2} = \frac{J_0}{J^2}, \quad T = 1
\]

(cf. T=0 is a MAP result)

Rujan (1993)
Nisimori (1993)
Nishimori and Wong (1998)
The Quantum version

\[
\hat{H} = \frac{Np^{-1}}{a^2 p!} \sum_{i1, i2, \ldots, ip} (J_{i1i2\ldots ip} - \frac{a_0 p!}{Np^{-1}} \hat{\sigma}_{i1} \cdots \hat{\sigma}_{ip})^2 - \Gamma \sum_i \hat{\sigma}_{ix}^2
\]

\[
\hat{\sigma}_{ix} = I_{(i)} \otimes \cdots \otimes \sigma_{ix} \otimes \cdots \otimes I_{(N)}
\]

\[
\hat{\sigma}_{iz} = I_{(i)} \otimes \cdots \otimes \sigma_{iz} \otimes \cdots \otimes I_{(N)}
\]

Density matrix

\[
\hat{\rho} = \frac{1}{Z} \exp\left[ -\frac{\hat{H}}{T} \right], \quad Z = \text{tr} \exp\left[ -\frac{\hat{H}}{T} \right]
\]

Bayes-optimal decoding is achieved even at the ground state \( T=0 \)

\[
\bar{\xi}_i = \text{sgn} \left[ \text{tr} \left( \hat{\sigma}_{iz} \hat{\rho} \right) \right]
\]

\[
\frac{a_0}{a^2} = \frac{J_0}{J^2}, \quad T \to 0, \quad \Gamma = \Gamma_{\text{opt}}
\]
Theoretical analysis (1/2)

In the large system limit $N \to \infty$

Phase diagram is independent of the amplitude of transverse field

$\xi_{i_1} \xi_{i_2} \cdots \xi_{i_p} = J_{i_1i_2\cdots i_p}^0$

$\frac{R}{C} = \left(\frac{J}{J_0}\right)^2 \log 2 \leq 1$

Infinite $p$

$\sqrt{\log 2} \approx 0.833$
Theoretical analysis (2/2)

Behavior of order parameters for finite $p$ in the large system limit

$p = 2: J_{ij} = \xi_i \xi_j$

$p = 3: J_{ijk} = \xi_i \xi_j \xi_k$

Parity bit

There exists the amplitude that gives the Bayes-optimal decoding

$\frac{a_0}{a^2} = \frac{J_0}{J^2} = 1$
Iterative decoding algorithm

For $p = 2$ we recursively solve

$$\hat{m}_i \equiv tr\left(\hat{\sigma}_i \hat{\rho}_{\text{mean-field}}\right)$$

$$m_i^{(t+1)} = \frac{\sum_j J_{ij}m_j^{(t)} - R_i^{(t)}}{\sqrt{\Gamma(t)^2 + \left(\sum_j J_{ij}m_j^{(t)} - R_i^{(t)}\right)^2}}$$

Reaction term

$$R_i^{(t)} = \sum_i J_{ij}^2 \left[2\left(1-\{m_i^{(t)}\}^2\right)\left(1-\{m_j^{(t)}\}^2\right)^{3/2} + 3\left(1-\{m_i^{(t)}\}^2\right)^{1/2}\left(1-\{m_j^{(t)}\}^2\right)^{3/2}\right]$$

with mean-field annealing

$$\Gamma(t) = \Gamma_0 \left(1 + \frac{c}{(t+1)}\right)$$

Bayes-estimate at each time step for each bit

$$\overline{\xi}_i^{(t)} = \text{sgn}(m_i^{(t)})$$

Estimate

$$\overline{\xi}_i = \text{sgn}\left[tr\left(\hat{\sigma}_i \hat{\rho}\right)\right]$$

TAP-like mean-field approximation

Is hard to compute
Dynamics of decoding

Practical decoding for finite system size

\( p = 2 \)
\( \Gamma_0 = 0.5 \)

N=1000, 10-samples

\( J_d/J = 0.8 \)
\( J_d/J = 1 \)
\( J_d/J = 2 \)
BER as a function of S/N ratio

Practical decoding for finite system size

\[ p = 2 \]
\[ \Gamma_0 = 0.5 \]

Phase transition
Summary

• We formulated quantum-mechanical decoding algorithm to retrieve the original message iteratively
• The BER drops suddenly at the critical signal-to-noise ratio
• *The Shannon’s bound* and *phase transition* are closely (exactly) related